

Singular Continuous Spectrum for the Laplacian on Certain Sparse Trees

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Abstract

We present examples of rooted tree graphs for which the Laplacian has singular continuous spectral measures. For some of these examples we further establish fractional Hausdorff dimensions. The singular continuous components, in these models, have an interesting multiplicity structure. The results are obtained via a decomposition of the Laplacian into a direct sum of Jacobi matrices.

1 Introduction

This note deals with the spectral analysis of the discrete Laplacian on trees that have a certain sparseness in their coordination number (to be precisely defined below). We show that the spectral theory of the Laplacian on such trees bears similarities to the theory of one-dimensional Schrödinger operators with a sparse-barrier potential. In particular, this framework allows us to construct explicit examples of trees with singular continuous spectrum. Moreover, for some of these models, the spectral measures have fractional Hausdorff dimensions (see theorem 4.2 below). Graphs with singular continuous spectrum are known to exist [10]; however, we are not aware of any previous explicit construction of a graph with non-trivial bounds on the spectral dimensions. What's more, we show that the singular continuous components occur with multiplicities that are related to the symmetry of the tree (theorem 2.5).

At the center of our analysis is a decomposition theorem (theorem 2.4) for the Laplacian on a family of trees that exhibit a certain spherical symmetry. All our examples follow from this decomposition by applying known methods from the theory of sparse-potential Schrödinger operators, mentioned above. Thus, for the applications, we content ourselves with giving the proper references and a few general remarks.

The paper is structured as follows. The basic result for sparse trees with singular continuous spectrum is described in section 2, along with the decomposition theorem. The proofs of theorems 2.4 and 2.5 are presented in Section 3. Further examples are given in section 4.

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2 Sparse Trees

Recall that for a combinatorial tree, the distance between two vertices is defined as the number of edges of the unique path between them.

Definition 2.1 (Spherically Homogeneous Rooted Tree). A rooted tree is called spherically homogeneous (SH) (see [2]) if any vertex v , at a distance j from the root - O , is connected with κ_j vertices at a distance $j + 1$ from O . A locally finite (that is - the valence of every vertex is finite) spherically homogeneous tree is uniquely determined by the sequence $\{\kappa_j\}_{j=0}^{\infty}$.

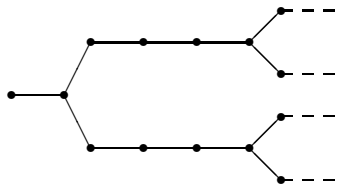


Figure 1: An example of a SH rooted tree with $\kappa_0 = 1$, $\kappa_1 = 2$, $\kappa_2 = \kappa_3 = \kappa_4 = 1$, $\kappa_5 = 2 \dots$

Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of natural numbers > 1 , and $\{L_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers. A spherically homogeneous

rooted tree - Γ - is said to be of type $\{L_n, k_n\}_{n=1}^\infty$, if

$$\kappa_j = \begin{cases} k_n & j = L_n \text{ for some } n \\ 1 & \text{otherwise} \end{cases} \quad (2.1)$$

We shall say that Γ is sparse if $(L_{n+1} - L_n) \rightarrow \infty$ rapidly, as $n \rightarrow \infty$.

Typical of the examples we construct is the following:

Theorem 2.2. *Let $k_0 \geq 2$ be a natural number and let $k_n \equiv k_0$. Assume that $(L_{n+1} - L_n) \rightarrow \infty$ and let Γ be a SH rooted tree, of type $\{L_n, k_n\}_{n=1}^\infty$. Then the essential spectrum of Δ on Γ contains the interval $[-2, 2]$ and, provided $(L_{n+1} - L_n)$ increase sufficiently rapidly, any spectral measure for Δ is purely singular continuous on $(-2, 2)$. By ‘sufficiently rapidly’ we mean that $(L_{n+1} - L_n)$ has to be made sufficiently large with respect to $\{(L_{i+1} - L_i)\}_{i < n}$.*

Note that since we are dealing with non-regular trees, there are two natural choices for the Laplacian:

$$(\Delta f)(x) = \sum_{y: d(x,y)=1} f(y), \quad (2.2)$$

and

$$(\tilde{\Delta} f)(x) = \sum_{y: d(x,y)=1} f(y) - \#\{y : d(x, y) = 1\} \cdot f(x) \quad (2.3)$$

where $\#A$, for a finite set A , is the number of elements in A ($d(x, y)$ denotes the distance on the tree). Although we formulate all our results for Δ , we note that they hold for $\tilde{\Delta}$ as well, (with $(-2, 2) \subseteq \mathbb{R}$ replaced by $(-4, 0)$ where necessary).

It is clear that if $\{k_n\}_{n=1}^\infty$ is a bounded sequence, then Δ on the tree is bounded and self-adjoint. For unbounded coordination number, the operator is unbounded and the issue of self-adjointness has to be addressed.

Definition 2.3. We call a SH rooted tree - Γ - *normal* if $\{k_n\}$ unbounded implies that $\limsup_{n \rightarrow \infty} (L_{n+1} - L_n) > 1$.

Standard methods imply that the Laplacian on normal SH rooted trees is self-adjoint.

The following decomposition theorem allows us to represent Δ on Γ as a direct sum of Jacobi matrices

$$J(\{a(j)\}_{j=1}^{\infty}, \{b(j)\}_{j=1}^{\infty}) = \begin{pmatrix} b(1) & a(1) & 0 & 0 & \dots \\ a(1) & b(2) & a(2) & 0 & \dots \\ 0 & a(2) & b(3) & a(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (2.4)$$

with

$$b(j) \in \mathbb{R}, \quad a(j) > 0.$$

Theorem 2.4. *Let Γ be a normal rooted SH tree of type $\{L_n, k_n\}_{n=1}^{\infty}$. Let*

$$M_n = \begin{cases} \prod_{j=1}^n k_j - \prod_{j=1}^{n-1} k_j & n > 1 \\ k_1 - 1 & n = 1 \\ 1 & n = 0. \end{cases} \quad (2.5)$$

Furthermore, let $R_0 = 0$ and $R_n = L_n + 1$, for $n \geq 1$. Then Δ is unitarily equivalent to a direct sum of Jacobi matrices, each operating on a copy of $\ell^2(\mathbb{Z}^+)$:

$$\Delta \cong \oplus_{n=0}^{\infty} \underbrace{(J_n \oplus J_n \oplus \dots \oplus J_n)}_{M_n \text{ times}} \quad (2.6)$$

where $J_n = J(\{a_n(j)\}_{j=1}^{\infty}, \{b_n(j)\}_{j=1}^{\infty})$ with

$$a_n(j) = \begin{cases} \sqrt{k_m} & j = R_m - R_n \text{ for some } m > n \\ 1 & \text{otherwise} \end{cases} \quad (2.7)$$

and

$$b_n(j) \equiv 0. \quad (2.8)$$

Remarks. 1. For the case of a regular tree, a similar decomposition was discussed in [1, 8] (see also [13] for a related result in the case of a metric tree).

2. As noted above, this theorem holds for $\tilde{\Delta}$ as well, albeit with different values for $\tilde{b}_n(j)$.

The next theorem is a simple corollary of theorem 2.4. Its conditions are satisfied by all the cases we consider in this paper.

Theorem 2.5. *Let Γ be a normal SH rooted tree of type $\{L_n, k_n\}_{n=1}^\infty$. Consider Δ on Γ . Let $I \subseteq \mathbb{R}$ be an interval such that all the spectral measures restricted to I are singular continuous. Let P_I be the spectral projection onto I (associated with Δ), and let M_n be as defined in (2.5). Then, $P_I(\ell^2(\Gamma))$ decomposes as a direct sum of invariant spaces, $\oplus_{n=0}^\infty \mathcal{H}'_n$, such that Δ , restricted to \mathcal{H}'_n , has uniform multiplicity M_n and the measure classes associated with the representation of Δ restricted to \mathcal{H}'_n are mutually disjoint.*

Remark. It is not hard to show that the numbers M_n are dimensions of certain irreducible representations of the symmetry group of Γ .

Theorem 2.4 makes it clear that the spectral analysis of the Laplacian on sparse trees reduces to that of Jacobi matrices with ‘bumps’ that are sparse along the subdiagonal and superdiagonal. Such matrices are analogous to discrete one-dimensional Schrödinger operators with potentials composed of sparse barriers. There is extensive literature on the spectral theory of such operators (see [5] for a review of the relevant theory), showing that such potentials give rise to a variety of interesting spectral phenomena. As noted in the introduction, all the examples we present in this paper are obtained by applying the (suitably modified) methods of the diagonal sparse case to the off diagonal case and using theorem 2.4. In particular, theorem 2.2 follows from theorem 2.4 by the methods in [7].

3 Decomposing the Laplacian

We start with some terminology and notation: We use the shorthand $|v| \equiv d(v, O)$. For any $v \in \mathcal{V}(\Gamma)$ we call *the forward subtree of v* - Γ_v , the subtree of Γ all of whose vertices, u , satisfy the following two conditions

1. $|v| \leq |u|$.
2. any vertex v' on the unique path connecting v and u satisfies $|v| \leq |v'|$.

We shall use $S_\Gamma(r) \equiv \{v \in \mathcal{V}(\Gamma) \mid |v| = r\}$.

Proof of Theorem 2.4. We shall decompose $\mathcal{H} = \ell^2(\Gamma)$ as a direct sum of spaces - $\oplus_{n=0}^\infty \mathcal{H}_n$, each invariant under Δ , such that Δ restricted to \mathcal{H}_n is unitarily equivalent to a direct sum of M_n copies of J_n . We shall describe this decomposition inductively.

We need a label for some of the vertices of Γ : At a distance R_n from the root there are $\alpha_n \equiv \prod_{j=1}^n k_j$ vertices. These are naturally divided into α_{n-1} groups of k_n vertices with common backward neighbor. We shall label the vertices on $S_\Gamma(R_n)$ by $\{v_{n,l}\}_{l=1}^{\alpha_n}$ where for each $m = 1, 2, \dots, \alpha_{n-1}$, the vertices $\{v_{n,l}\}_{l=(m-1)k_n+1}^{mk_n}$ are all on the forward subtree of $v_{n-1,m}$.

In order to streamline the notation, we shall use $\delta_{n,l}$ for $\delta_{v_{n,l}}$ (the delta function at the vertex $v_{n,l}$). We shall also use $\Gamma_{n,l}$ for $\Gamma_{v_{n,l}}$.

Now, let us define $V_n = [\delta_{n,l}]_{l=1}^{\alpha_n}$ - the linear span of $\{\delta_{n,l}\}_{l=1}^{\alpha_n}$ - so that $\dim V_n = \alpha_n$. Let $\varphi_0 = \delta_O$ and let

$$\mathcal{H}_0 = \overline{[\Delta^n \varphi_0 \mid n = 0, 1, \dots]} \quad (3.1)$$

where $\overline{[\cdot]}$ for a linear subspace of \mathcal{H} denotes its closure. An orthogonal basis for \mathcal{H}_0 is obtained by ‘Gram-Schmidting’ the basis $\{\Delta^n \varphi_0\}$ - which results with normalized, radially symmetric functions supported on spheres around O (the radial symmetry is a consequence of the spherical homogeneity of Γ). This implies that \mathcal{H}_0 is the subspace of radially symmetric functions.

Now assume that we have defined \mathcal{H}_n for $n = 0, 1, \dots, (N-1)$ such that:

1. $\mathcal{H}_i \perp \mathcal{H}_j$ for any $i \neq j$ and all spaces are invariant under Δ in the sense that $\Delta(D(\Delta) \cap \mathcal{H}_i) \subseteq \mathcal{H}_i$ (where $D(\Delta)$ is the domain of Δ).
2. For any vertex v with $|v| < R_N$, $\delta_v \in \oplus_{n=1}^{(N-1)} \mathcal{H}_n$.
3. For any $l > (N-1)$, $V_l \cap \left(\oplus_{n=1}^{(N-1)} \mathcal{H}_n\right)$ is $\alpha_{(N-1)}$ dimensional.

(All these properties hold trivially for $N-1 = 0$ if we let $\alpha_0 = 1$). Recall that $M_n = \prod_{j=1}^n k_j - \prod_{j=1}^{n-1} k_j = \alpha_n - \alpha_{n-1}$. From the above, it follows that the orthogonal complement in V_N (which is α_N dimensional) to $\oplus_{n=1}^{(N-1)} \mathcal{H}_n$ is M_N dimensional and is spanned by M_N mutually orthogonal unit vectors - $\varphi_{N,1}, \dots, \varphi_{N,M_N}$. Writing

$$\varphi_{N,j} = \sum_{l=1}^{\alpha_N} a_{N,j}^l \delta_{N,l}, \quad 1 \leq j \leq M_N \quad (3.2)$$

and recalling that for all $m = 1, 2, \dots, \alpha_{(N-1)}$, $\{v_{N,l}\}_{l=(m-1)k_N+1}^{mk_N}$ have a common backward neighbor on $S_\Gamma(L_N)$, we get (from 2 above) that

$$\sum_{l=(m-1)k_N+1}^{mk_N} a_{N,j}^l = 0 \quad (3.3)$$

for all m and j . Define

$$\mathcal{H}_N = \overline{[\Delta^n \varphi_{N,j} \mid 1 \leq j \leq M_N \ n = 0, 1, \dots]}. \quad (3.4)$$

Then we claim that

$$\mathcal{H}_N = \oplus_{j=1}^{M_N} \mathcal{H}_{N,j} \quad (3.5)$$

with

$$\mathcal{H}_{N,j} = \overline{[\Delta^n \varphi_{N,j} \mid n = 0, 1, \dots]}. \quad (3.6)$$

Indeed, (3.3) together with the spherical homogeneity of Γ , implies that the orthogonal basis obtained from the Gram-Schmidt process applied to $\{\Delta^n \varphi_{N,j}\}$, is made of functions supported on $\{v \mid |v| \geq R_N\}$ and having the form

$$\frac{1}{\rho_{N,j}(r)} \sum_{|v|=r} a_{N,j}^v \delta_v \quad (3.7)$$

where $r \geq R_N$, $\rho_{N,j}(r) > 0$ is a normalizing factor and $a_{N,j}^v = a_{N,j}^l$ if $v \in \Gamma_{N,l}$. Together with the orthogonality of the various $\varphi_{N,j}$, this means that $\mathcal{H}_{N,j} \perp \mathcal{H}_{N,i}$ if $i \neq j$. Thus we see that properties 1-3 above hold for $n = 0, 1, \dots, N$.

Having constructed \mathcal{H}_n for all n in this fashion, we get from properties 1 and 2 that indeed

$$\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n \quad (3.8)$$

and that each of the spaces in the direct sum is invariant under Δ in the sense that $\Delta(D(\Delta) \cap \mathcal{H}_n) \subseteq \mathcal{H}_n$. This almost means that Δ decomposes as a direct sum. The only missing point is that if Δ is unbounded, the above does not necessarily mean that \mathcal{H}_n is invariant under $(\Delta - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. However, it is easy to see that the moment problem associated with the operation of Δ on $\varphi_{n,j}$ (any n and $1 \leq j \leq M_n$) is determinate, so it follows from proposition 4.15 in [11] that \mathcal{H}_n is indeed invariant in both senses. Thus, we have that

$$\Delta = \oplus_{n=1}^{\infty} \Delta_n \quad (3.9)$$

with Δ_n denoting the corresponding restricted operators.

In order to complete the proof, we need to show that

$$\Delta_n \cong \underbrace{J_n \oplus J_n \oplus \dots \oplus J_n}_{M_n \text{ times}}. \quad (3.10)$$

Knowing (3.5) and (3.7), however, this is now a matter of simple computation (since it is easy to see that $\rho_{n,j}(r) = \sqrt{\#(S_\Gamma(r) \cap \Gamma_{n,l})}$ for any $1 \leq l \leq \alpha_n$ and $r \geq R_n$).

□

Proof of Theorem 2.5. Theorem 2.4 says that it suffices to consider the measures μ_n - the spectral measures of J_n and $\delta_1 \in \ell^2(\mathbb{Z}^+)$ (since this is a cyclic vector). Obviously, these measures occur with multiplicity at least M_n , so we only need to show that their singular continuous parts are mutually singular. Note now that for $n_1 > n_2$, J_{n_1} can be obtained from J_{n_2} by ‘stripping off’ the $(R_{n_1} - R_{n_2})$ leftmost columns and the same number of rows from the top. The fact that the singular continuous part of μ_{n_1} is singular, with respect to the singular continuous part of μ_{n_2} , follows, now, from the characterization of the appropriate supports in terms of m -functions (see e.g. [9]) and from the continued fraction expansion of m [3]. (The spaces \mathcal{H}'_n are just $P_I(\mathcal{H}_n)$). □

4 Singular Continuous Spectrum for Sparse Trees

Let Γ be a sparse, normal SH rooted tree. As noted in the proof of theorem 2.5, all the matrices J_n , in the decomposition of the Laplacian, are actually various ‘tails’ of J_0 . Since, in the sparse models, it is the asymptotics that determine the spectral type, this means that the spectral analysis of the Laplacian reduces essentially to the analysis of a single Jacobi matrix.

A good reason for considering trees with unbounded $\{k_n\}$, is the fact that for trees with $k_n \rightarrow \infty$, it is easy to state explicit growth conditions on $\{L_n\}$ which make the spectrum singular continuous. In particular, $k_n \rightarrow \infty$ implies absence of absolutely continuous spectrum [6], so the following is a straightforward adaptation of an idea of Simon-Stolz [12] to our case:

Theorem 4.1. *Let $\{k_n\}_{n=1}^\infty$ be a sequence of natural numbers such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $\alpha_n = \prod_{j=1}^n k_j$. Assume that $(L_{n+1} - L_n) \rightarrow \infty$ and let Γ be a SH rooted tree of type $\{L_n, k_n\}_{n=1}^\infty$. Then the spectrum of Δ on Γ consists of the interval $[-2, 2]$ along with some discrete point spectrum outside this interval. If for some $\varepsilon > 0$,*

$$\limsup_{n \rightarrow \infty} \frac{(L_{n+1} - L_n)}{\alpha_n^{(1+\varepsilon)}} > 0, \quad (4.1)$$

then any spectral measure for Δ is purely singular continuous on $(-2, 2)$.

As certain sparse potentials have been constructed with spectral measures of fractional Hausdorff dimensionality, it seems natural to try to construct trees with this property as well. An adaptation of an example of Jitomirskaya-Last [4] (see also [14]) achieves just that:

Theorem 4.2. *Let $L_n = 2^{(n^n)}$. Let $\beta > 0$ and $k_n = \lfloor L_n^\beta \rfloor$. Let Γ_β be the corresponding tree. Then the restriction to $(-2, 2)$ of any spectral measure for Δ on Γ_β , is supported on a set of Hausdorff dimension $\frac{2}{2+\beta}$ and does not give weight to sets of Hausdorff dimension less than $\frac{1}{1+\beta}$.*

Letting $k_n = \lfloor L_n^{\beta_n(c)} \rfloor$ with $\beta_n(c) = c \frac{(n+1)^{(n+1)}}{n^n}$ for some $1 > c > 0$, we get that any spectral measure on $(-2, 2)$ is purely singular continuous and supported on a set of Hausdorff dimension 0.

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